

Pancyclicity when each cycle contains k chords

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Abstract

For integers $n \geq k \geq 2$, let $c(n, k)$ be the minimum number of chords that must be added to a cycle of length n so that the resulting graph has the property that for every $l \in \{k, k+1, \dots, n\}$, there is a cycle of length l that contains exactly k of the added chords. Chaouche, Rutherford, and Whitty introduced the function $c(n, k)$. They showed that for every integer $k \geq 2$, $c(n, k) \geq \Omega_k(n^{1/k})$ and they asked if $n^{1/k}$ gives the correct order of magnitude of $c(n, k)$ for $k \geq 2$. Our main theorem completely answers this question as we prove that for every integer $k \geq 2$, $c(n, k) \leq k \lceil n^{1/k} \rceil + k$. This upper bound, together with the lower bound of Chaouche et. al., shows that the order of magnitude of $c(n, k)$ is $n^{1/k}$.

1 Introduction

An n -vertex graph is **pancyclic** if it contains a cycle of length l for each $l \in \{3, 4, 5, \dots, n\}$. There is a large amount of research on pancyclic graphs including many papers on conditions which imply pancyclicity, as well as different properties of pancyclic graphs. For more on pancyclic graphs, we refer the reader to the recent textbook of George, Khodkar, and Wallis [4]. Our focus will be on an extremal function that has its roots in a function

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of Bondy. Let $m(n)$ be the minimum number of edges in a pancyclic graph on n vertices. Bondy [3] introduced the function $m(n)$ and stated, without proof, that

$$n - 1 + \log_2(n - 1) \leq m(n) \leq n + \log_2 n + H(n) + O(1) \quad (1)$$

where $H(n)$ is the smallest integer such that $(\log_2)^{H(n)}(n) < 2$. To our knowledge, the bounds given in (1) have not been improved. Nevertheless, Bondy's problem is quite natural and has inspired several new extremal functions. Let us take a moment to introduce one such example before we define the extremal function that we will focus on.

Broersma [1] asked for the minimum number of edges in a vertex pancyclic graph on n vertices. Recall an n -vertex graph is **vertex pancyclic** if every vertex lies in a cycle of length l for every $l \in \{3, 4, 5, \dots, n\}$. Broersma proved that

$$\frac{3}{2}n < vp(n) \leq \lfloor \frac{5}{3}n \rfloor \quad (2)$$

for all $n \geq 7$ where $vp(n)$ is the minimum number of edges in a vertex pancyclic graph with n vertices. This shows that $vp(n)$ is linear in n , but an asymptotic formula for $vp(n)$ is not known. Improving either (1) or (2) would be quite interesting.

A consequence of (1) is that if $m'(n)$ is the minimum number of chords that must be added to C_n to obtain a pancyclic graph, then $m'(n) = \log_2 n + o(\log_2 n)$. Similarly, (2) implies that the minimum number of chords that must be added to C_n to obtain a vertex pancyclic graph is $\Theta(n)$. Motivated by these results, Chaouche, Rutherford, and Whitty [2] introduced the following extremal function. For integers $n \geq 6$ and $k \geq 2$, let $c(n, k)$ be the minimum number of chords that must be added to a cycle of length n so that the resulting graph has the property that for every integer $l \in \{k, k + 1, \dots, n\}$, there is a cycle of length l that contains exactly k of the added chords. To avoid considering cases when $k = 2$ and $k \geq 3$ separately, we count a closed walk of the form xyx as a cycle of length 2. By (1), we immediately get $c(n, k) \geq \log_2(n - 1)$ for all $k \geq 2$. The following result of Chaouche et. al. improves this lower bound and shows that requiring each cycle to contain exactly k of the added chords has a rather dramatic effect on the amount of chords that must be added.

Theorem 1 (Chaouche, Rutherford, Whitty [2]). *Let $k \geq 1$ be an integer. For any integer $n \geq 6$, there is a constant α_k , depending only on k , such that*

$$c(n, k) \geq \alpha_k n^{1/k}.$$

In [2], it is suggested that $c(n, k)$ lies somewhere between $m'(n) = \Theta(\log n)$ and $\Theta(n)$. It was left as an unsolved problem to prove a non-trivial upper bound on $c(n, k)$. Our main result solves this problem completely, and together with Theorem 1, shows that the order of magnitude of $c(n, k)$ is $n^{1/k}$.

Theorem 2. *Let $k \geq 2$ be an integer. If $n \geq (k + 2)^k$, then*

$$c(n, k) \leq k \lceil n^{1/k} \rceil + k.$$

A close look at the proof of Theorem 1 shows that the constant α_k is asymptotic to k/e as k goes to infinity. Therefore, our Theorem 2 is best possible up to a constant factor. In fact, we believe that for $k \geq 2$, $c(n, k) = kn^{1/k} + o(n^{1/k})$ and Theorem 2 is asymptotically best possible.

In the next section we prove Theorem 2. Some concluding remarks are made in Section 3. Most of the notation that we use follows that of West [6]. In particular, C_n always denotes a cycle of length n with vertex set $\{1, 2, \dots, n\}$ whose edges are $\{i, i+1\}$, $i \in \{1, 2, \dots, n\}$ together with $\{n, 1\}$.

2 Proof of Theorem 2

Let $k \geq 2$ be an integer and let $n \geq (k + 2)^k$. In order to prove that $c(n, k) \leq k \lceil n^{1/k} \rceil + k$, we must show how to add at most $k \lceil n^{1/k} \rceil + k$ chords to C_n so that in the resulting graph, for each $l \in \{k, k+1, \dots, n\}$, there is a cycle of length l that contains exactly k chord edges. The construction is best described in several steps.

Definition 3. Let $b \geq 3$, $i \geq 1$, and $e \geq 0$ be integers. Let $G_b(i, e)$ be the graph with vertex set

$$\{i, i+1, i+2, \dots, i+2+b^{e+1}-b^e\}$$

and edge set

$$\{\{j, j+1\} : i \leq j \leq i+1+b^{e+1}-b^e\} \cup \{\{i, i+2+jb^e\} : 0 \leq j \leq b-1\}.$$

The vertex i is called the **base point** of $G_b(i, e)$. Edges in the set

$$\{\{i, i + 2 + jb^e\} : 0 \leq j \leq b - 1\}$$

are called **base chords**. Edges of the form $\{j, j + 1\}$ are called **outer edges**.

Observe that in $G_b(i, e)$ there are exactly b base chords and exactly $b^{e+1} - b^e + 2$ outer edges.

Lemma 4. *For each $c_e \in \{0, 1, \dots, b - 1\}$, the graph $G_b(b^e + 2e, e)$ contains a path P of length $1 + c_e b^e$ from the vertex $b^e + 2e$ to the vertex $b^{e+1} + 2(e + 1)$ such that P contains exactly one base chord.*

Proof of Lemma 4. Let $c_e \in \{0, 1, 2, \dots, b - 1\}$ and define $x = b^{e+1} + 2(e + 1) - c_e b^e$. Consider the path P whose first edge is $\{b^e + 2e, x\}$, and whose remaining edges are

$$\{\{x + j, x + j + 1\} : 0 \leq j \leq c_e b^e - 1\}.$$

The path P has $1 + c_e b^e$ edges. The only base chord in the path is $\{b^e + 2e, x\}$. To show that this edge is in fact a base chord, note that

$$x = b^{e+1} + 2(e + 1) - c_e b^e = (b^e + 2e) + 2 + (b - c_e - 1)b^e$$

and $b - c_e - 1 \in \{0, 1, \dots, b - 1\}$ since $c_e \in \{0, 1, \dots, b - 1\}$. \square

The graphs $G_b(b^e + 2e, e)$ form the building blocks of our construction and they will be put together using the graph union operation. Recall that if G_1, \dots, G_r is a collection of graphs, then the graph $G_1 \cup G_2 \cup \dots \cup G_r$ is the graph with vertex set

$$V(G_1) \cup V(G_2) \cup \dots \cup V(G_r)$$

and edge set

$$E(G_1) \cup E(G_2) \cup \dots \cup E(G_r).$$

Definition 5. For integers $k \geq 2$ and $b \geq 3$, let $H_b(k)$ be the graph

$$H_b(k) = \bigcup_{e=0}^{k-1} G_b(b^e + 2e, e).$$

For example, the graph $H_4(3)$ is the union of the three graphs $G_4(1, 0)$, $G_4(6, 1)$, and $G_4(20, 2)$ which are shown in Figure 1. In the figure, all of the edges are shown, but not all of the vertices of $G_4(6, 1)$ and $G_4(20, 2)$ are shown.

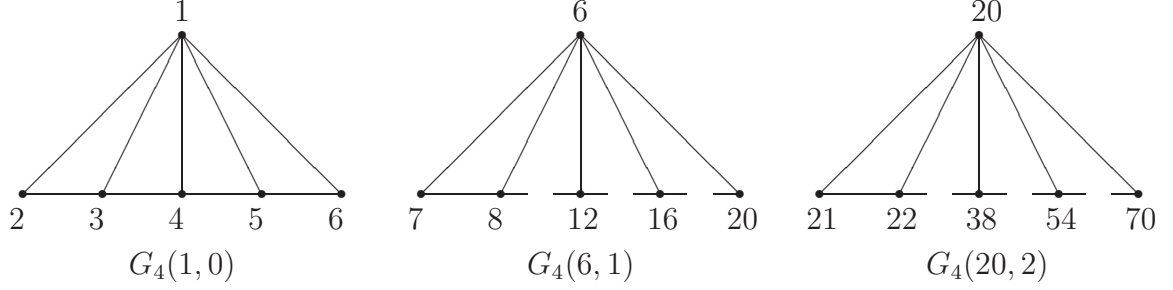


Figure 1: The graphs $G_4(1, 0)$, $G_4(6, 1)$, and $G_4(20, 2)$.

For $0 \leq e_1 < e_2 \leq k - 1$, the intersection

$$V(G_b(b^{e_1} + 2e_1, e_1)) \cap V(G_b(b^{e_2} + 2e_2, e_2))$$

is empty unless $e_2 = e_1 + 1$ in which case the intersection is precisely $\{b^{e_1+1} + 2(e_1 + 1)\}$. This implies that the graphs

$$G_b(1, 0), G_b(b + 2, 1), G_b(b^2 + 4, 2), \dots, G_b(b^{k-1} + 2(k - 1), k - 1)$$

are edge disjoint. Therefore, $H_b(k)$ has $b^k + 2k$ vertices and $b^k + 2k - 1 + bk$ edges. An edge of $H_b(k)$ that is a base chord of some $G_b(b^e + 2e, e)$ is called a **chord edge**. As each $G_b(b^e + 2e, e)$ has b base chords, the graph $H_b(k)$ has bk chord edges. The edges of $H_b(k)$ that are not chord edges form a path from the vertex 1 to the vertex $b^k + 2k$.

The next lemma is key to our construction. It shows how to find long paths in the graph $H_b(k)$ that pass through a specified number of chord edges. These paths will be used later to construct almost all of the necessary cycles.

Lemma 6. *Let c_0, c_1, \dots, c_{k-2} be arbitrary elements of $\{0, 1, \dots, b - 1\}$. In the graph $H_b(k)$, there is a path of length*

$$k - 1 + \sum_{e=0}^{k-2} c_e b^e$$

from the vertex 1 to the vertex $b^{k-1} + 2(k - 1)$ that contains exactly $k - 1$ chord edges.

Proof of Lemma 6. Let $e \in \{0, 1, \dots, k-2\}$. By Lemma 4, there is a path P_e of length $1 + c_e b^e$ in the unique copy of $G_b(b^e + 2e, e)$ in $H_b(k)$ where the first vertex of P_e is $b^e + 2e$, the last vertex is $b^{e+1} + 2(e+1)$, and P_e contains exactly one chord edge. Therefore, the union

$$P_0 \cup P_1 \cup \dots \cup P_{k-2}$$

is a path of length $k-1 + \sum_{e=0}^{k-2} c_e b^e$ from the vertex 1 to the vertex $b^{k-1} + 2(k-1)$ that contains exactly $k-1$ chord edges. \square

We will require another definition in order to continue the description of the construction.

Definition 7. Let $k \geq 2$ and $n \geq (k+2)^k$ be integers. Define $b = \lceil n^{1/k} \rceil$. Let $F_k(n)$ be the graph obtained by adding the edges

$$\{1, b^{k-1} + 2(k-1)\} \text{ and } \{1, n\}$$

to the subgraph of $H_b(k)$ induced by the vertices $\{1, 2, \dots, n\}$.

The edge $\{1, b^{k-1} + 2(k-1)\}$ is called a **chord edge** and all of the chord edges in $H_b(k)$ are also called **chord edges** in $F_k(n)$.

Observe that the number of chord edges of $F_k(n)$ is at most $k \lceil n^{1/k} \rceil + 1$ and that $F_k(n)$ contains a cycle of length n whose edges are

$$\{1, n\} \cup \{\{i, i+1\} : 1 \leq i \leq n-1\}.$$

The number of total chord edges is what we are interested in counting to give an upper bound for $c(n, k)$. The next lemma will show that for all $l \in \{k, k+1, \dots, n-k\}$, we can find a cycle of length l in $F_k(n)$ that contains exactly k chords. Therefore, the graph $F_k(n)$ contains almost all of the cycles that we need in order to complete the proof of Theorem 2.

Lemma 8. *Let $k \geq 2$ and $n \geq (k+2)^k$. For each $l \in \{k, k+1, \dots, n-k\}$, the graph $F_k(n)$ contains a cycle of length l that passes through exactly k chord edges.*

Proof of Lemma 8. We will prove Lemma 8 by establishing several claims.

Claim 9. *We have $n > b^{k-1} + 2k$ and $(b-1)^k < n \leq b^k$.*

Proof of Claim 9. Since $b = \lceil n^{1/k} \rceil$, we immediately have $(b-1)^k < n \leq b^k$.

If $n \geq (k+2)^k$, then $b \geq k+2$. Since we claim that $n > b^{k-1} + 2k$ and $n > (b-1)^k$, it is sufficient to show that $(b-1)^k > b^{k-1} + 2k$. Since $(b-1)^k$ grows more quickly than b^{k-1} , showing that the inequality is true for the smallest case, $b = k+2$ and $k = 2$, means it is true for all $b \geq k+2$ and $k \geq 2$. Plugging in the appropriate values we get $(k+1)^2 > (k+2) + 4$ and $9 > 8$. \square

By Definition 7, $F_k(n)$ contains exactly one copy of each of the graphs

$$G_b(1, 0), G_b(b+2, 1), G_b(b^2+4, 2), \dots, G_b(b^{k-2}+2(k-2), k-2).$$

The vertex $b^{k-1} + 2(k-1)$, which is the unique vertex in both $G_b(b^{k-2} + 2(k-2), k-2)$ and $G_b(b^{k-1} + 2(k-1), k-1)$, will play a special role so we let

$$m = b^{k-1} + 2(k-1).$$

Since $V(G_b(m, k-1)) = \{m, m+1, m+2, \dots, b^k + 2k\}$, the graph $F_k(n)$ contains a non-trivial induced subgraph of $G_b(m, k-1)$. Now we will define an integer α that counts the number of base chords in $G_b(m, k-1)$ that are chord edges in $F_k(n)$. Let α be the unique integer in the set $\{0, 1, \dots, b-2\}$ that satisfies

$$m + 2 + \alpha b^{k-1} \leq n < m + 2 + (\alpha + 1)b^{k-1}. \quad (3)$$

Such an α exists by Claim 9. The base chords in $G_b(m, k-1)$ that are chords in $F_k(n)$ are precisely those edges in the set

$$\{\{m, m+2+jb^{k-1}\} : j \in \{0, 1, \dots, \alpha\}\}.$$

By Lemma 6, for any $c_0, c_1, \dots, c_{k-2} \in \{0, 1, \dots, b-1\}$, there is a path, which we will denote by $P(c_0, c_1, \dots, c_{k-2})$, that has the following properties.

1. The first vertex is 1 and the last vertex is m .
2. The length of $P(c_0, c_1, \dots, c_{k-2})$ is $k-1 + \sum_{e=0}^{k-2} c_e b^e$.
3. The edges of $P(c_0, c_1, \dots, c_{k-2})$ are edges in the graphs

$$G_b(1, 0), G_b(b+2, 1), G_b(b^2+4, 2), \dots, G_b(b^{k-2}+2(k-2), k-2).$$

4. The path $P(c_0, c_1, \dots, c_{k-2})$ contains exactly $k-1$ chord edges of $F_k(n)$.

We will now find paths in $F_k(n)$ from the vertex m to the vertex 1 that use edges from $G_b(m, k-1)$, and either $\{n, 1\}$ or $\{m, 1\}$. Each of these paths will also contain exactly one chord edge.

Claim 10. *For every $l \in \{k, k+1, \dots, k+b^{k-1}-1\}$ the graph $F_k(n)$ has a cycle of length l that contains exactly k chord edges.*

Proof of Claim 10. By definition of $F_k(n)$, the edge $\{m, 1\}$ is a chord edge. Therefore, given $c_0, c_1, \dots, c_{k-2} \in \{0, 1, \dots, b-1\}$, the union

$$P(c_0, c_1, \dots, c_{k-2}) \cup \{m, 1\}$$

is a cycle of length $k + \sum_{e=0}^{k-2} c_e b^e$ that contains exactly k chord edges. Claim 10 now follows from the fact that every integer in the set $\{0, 1, \dots, b^{k-1}-1\}$ can be written in the form $\sum_{e=0}^{k-2} c_e b^e$ for some $c_0, c_1, \dots, c_{k-2} \in \{0, 1, \dots, b-1\}$. \square

Let

$$L_1 = \left\{ k + \sum_{e=0}^{k-2} c_e b^e : c_e \in \{0, 1, \dots, b-1\} \right\} = \{k, k+1, \dots, k+b^{k-1}-1\}.$$

Claim 11. *For any $j \in \{0, 1, \dots, \alpha\}$, there is a path $Q(j)$ in $F_k(n)$ with the following properties.*

1. *The first vertex is m and the last vertex is 1.*
2. *The length of $Q(j)$ is $n+2-(j+1)b^{k-1}-2k$.*
3. *All of the edges of $Q(j)$, with the exception of $\{n, 1\}$, are edges in the partial copy of $G_b(m, k-1)$ in $F_k(n)$.*
4. *The path $Q(j)$ contains exactly one chord edge.*

Proof of Claim 11. Let $j \in \{0, 1, \dots, \alpha\}$ and let $Q(j)$ be the path

$$m, m+2+jb^{k-1}, m+2+jb^{k-1}+1, m+2+jb^{k-1}+2, \dots, n-1, n, 1.$$

Clearly the first property holds for $Q(j)$. The length of $Q(j)$ is

$$1 + (n+1) - (m+2+jb^{k-1}) = n+2 - (b^{k-1} + 2(k-1) + 2 + jb^{k-1}).$$

This shows that the second property holds. The third property follows from the definition of the graphs $G_b(i, e)$. Finally, the fourth property follows from the fact that the first edge of $Q(j)$, which is $\{m, m + 2 + jb^{k-1}\}$, is the only chord edge in $Q(j)$. \square

Given $c_0, c_1, \dots, c_{k-2} \in \{0, 1, \dots, b-1\}$ and $j \in \{0, 1, \dots, \alpha\}$, the union

$$P(c_0, c_1, \dots, c_{k-2}) \cup Q(j)$$

is a cycle in $F_k(n)$ of length

$$k - 1 + \sum_{e=0}^{k-2} c_e b^e + n + 2 - (j + 1)b^{k-1} - 2k$$

that contains exactly k chords. This expression can be rewritten as

$$n + 1 - k - (j + 1)b^{k-1} + \sum_{e=0}^{k-2} c_e b^e.$$

Let

$$L_2 = \{n + 1 - k - (j + 1)b^{k-1} + \sum_{e=0}^{k-2} c_e b^e : j \in \{0, 1, \dots, \alpha\}, c_e \in \{0, 1, \dots, b-1\}\}.$$

Claim 12. *If L_2 is defined as above and $\gamma = n - k + 1$, then*

$$\{\gamma - (\alpha + 1)b^{k-1}, \gamma - (\alpha + 1)b^{k-1} + 1, \gamma - (\alpha + 1)b^{k-1} + 2, \dots, n - k\} \subseteq L_2.$$

Proof of Claim 12. It is easy to check that the smallest integer in L_2 is

$$n - k + 1 - (\alpha + 1)b^{k-1} = \gamma - (\alpha + 1)b^{k-1}$$

and the largest integer in L_2 is

$$n - k + b^{k-1} - b^{k-1} = n - k.$$

We will now show that L_2 contains every integer between $\gamma - (\alpha + 1)b^{k-1}$ and $n - k$.

As the c_e 's range over $\{0, 1, \dots, b-1\}$, the sum $\sum_{e=0}^{k-2} c_e b^e$ ranges over all integers in the set $\{0, 1, \dots, b^{k-1} - 1\}$. Thus, for any fixed $j \in \{0, 1, \dots, \alpha\}$, the set

$$I_j = \{\gamma - (j+1)b^{k-1} + \sum_{e=0}^{k-2} c_e b^e : c_e \in \{0, 1, \dots, b-1\}\}$$

is the interval

$$\{\gamma - (j+1)b^{k-1}, \gamma - (j+1)b^{k-1} + 1, \dots, \gamma - jb^{k-1} - 1\}.$$

Note that for $0 < j \leq \alpha$, the largest integer in I_j is $\gamma - jb^{k-1} - 1$ and the smallest integer in I_{j+1} is $\gamma - jb^{k-1}$. These two integers are consecutive and so the union

$$I_0 \cup I_1 \cup \dots \cup I_\alpha$$

contains all integers from $\gamma - (\alpha+1)b^{k-1}$ to $\gamma - 0 \cdot b^{k-1} - 1 = n - k$. This completes the proof of Claim 12. \square

Claim 13. *We have*

$$\{k, k+1, \dots, n-k\} \subseteq L_1 \cup L_2.$$

Proof of Claim 13. To prove Claim 13, it is enough to show that

$$n+1-k-(\alpha+1)b^{k-1}-1 \leq k+b^{k-1}-1 \quad (4)$$

since the largest integer in L_1 is $k+b^{k-1}-1$, and the smallest integer in L_2 is $n+1-k-(\alpha+1)b^{k-1}$. The inequality (4) is equivalent to

$$n+1 \leq (\alpha+2)b^{k-1} + 2k. \quad (5)$$

Recalling the definition of α given in (3), we have that $n < (\alpha+2)b^{k-1} + 2k$. Now since n and $(\alpha+2)b^{k-1} + 2k$ are integers, this last inequality implies (5). \square

Combining Claim 13 together with the fact that for each $l \in L_1 \cup L_2$, $F_k(n)$ contains a cycle of length l with exactly k chord edges completes the proof of Lemma 8. \square

The final step in the construction is to add an additional $k-1$ chord edges to $F_k(n)$ to account for the cycle lengths in the set $\{n-k+1, n-k+2, \dots, n\}$.

Definition 14. Let $D_2(n)$ be the graph obtained by adding the edge $\{2, b+3\}$ to $F_2(n)$. For $k \geq 3$, let $D_k(n)$ be the graph obtained by adding the edge $\{b^{k-2} + 2(k-2) + 1, m+1\}$ and the edges $\{b^e + 2e + 1, b^{e+2} + 2(e+2)\}$ for $e \in \{0, 1, \dots, k-3\}$ to $F_k(n)$.

The edges added to $F_k(n)$ to obtain $D_k(n)$ are called **chord edges**. Any chord edge of $F_k(n)$ is also called a chord edge of $D_k(n)$. All other edges of $D_k(n)$ are outer edges and these edges form an n -cycle C_n .

Lemma 15. *The graph $D_k(n)$ contains a cycle of length l for each $l \in \{k, k+1, \dots, n\}$ that contains exactly k chord edges.*

Proof of Lemma 15. We will prove Lemma 15 by proving two claims about $D_k(n)$.

Claim 16. *There is a path S in $D_k(n)$ from the vertex 2 to the vertex 1 that has length $n - (b+1)$, does not use any outer edges in the set $\{\{i, i+1\} : i \in \{1, 2, \dots, b+1\}\}$, and has $k-1$ chord edges, specifically the chord edges in $D_k(n) \setminus F_k(n)$.*

Proof of Claim 16. We will consider two cases depending on k .

If $k = 2$, the first edge of S is $\{2, b+3\}$ and the remaining edges are $\{b+3+i, b+3+i+1\}$ for $0 \leq i \leq n - (b+4)$. The last edge of S is $\{n, 1\}$.

Now assume $k \geq 3$. In this instance S is best defined by a union of three edge-disjoint subpaths which we will denote by S_1, S_2, S_3 . The chord edges in S_1 will be of the form

$$\{b^e + 2e + 1, b^{e+2} + 2(e+2)\}.$$

The first edge of S_1 is $\{2, b^2 + 4\}$. From the vertex $b^2 + 4$, the next segment of S_1 is the path $b^2 + 4, b^2 + 4 - 1, b^2 + 4 - 2, \dots, b + 3$. Observe that this segment of S_1 contains only outer edges. Up to the vertex $b + 3$, S_1 has a length of $1 + (b^2 - b + 1)$. Next S_1 contains the chord edge $\{b + 3, b^3 + 6\}$, followed by the path $b^3 + 6, b^3 + 6 - 1, b^3 + 6 - 2, \dots, b^2 + 4 + 1$ which consists only of outer edges.

This pattern is continued for $e = 2, 3, \dots, k-4$ by taking the chord edge $\{b^e + 2e + 1, b^{e+2} + 2(e+2)\}$ followed by the path of outer edges $b^{e+2} + 2(e+2), b^{e+2} + 2(e+2) - 1, \dots, b^{e+1} + 2(e+1) + 1$. The last edge of S_1 is the chord edge $\{b^{k-3} + 2(k-3), b^{k-1} + 2(k-1)\}$.

Therefore the amount of chord edges contained in S_1 is $k - 2$ and S_1 ends at the vertex $m = b^{k-1} + 2(k - 1)$. Up to vertex m , S_1 has length of

$$k - 2 + \sum_{e=1}^{k-3} (b^{e+1} - b^e + 1)$$

which can be rewritten as

$$2(k - 2) - 1 + b^{k-2} - b.$$

The path S_2 begins with vertex m and contains edges $\{m - j, m - j - 1\}$ for $j \in \{0, 1, 2, \dots, b^{k-1} - b^{k-2}\}$ together with the chord edge $\{b^{k-2} + 2(k - 2) + 1, m + 1\}$. The length of S_2 is $(b^{k-1} - b^{k-2} + 1) + 1$ and the length of $S_1 \cup S_2$ is

$$2(k - 2) - 1 + b^{k-2} - b + (b^{k-1} - b^{k-2} + 1) + 1 = 2(k - 1) - 1 + b^{k-1} - b.$$

Beginning with the vertex $m + 1$, S_3 contains all edges $\{m + 1 + j, m + 1 + j + 1\}$ for $j \in \{0, 1, \dots, n - m - 2\}$ together with $\{n, 1\}$. We then take $S = S_1 \cup S_2 \cup S_3$, which starts with vertex 2 and ends at vertex 1, contains $k - 1$ chords, and has a length of

$$2(k-1)-1+b^{k-1}-b+n-m = 2(k-1)-1+b^{k-1}-b+n-(2(k-1)+b^{k-1}) = n-b-1.$$

□

Claim 17. *For each $a \in \{1, 2, \dots, b\}$, there is a path T from vertex the 1 to the vertex 2 of length $a + 1$ that contains exactly one chord.*

Proof of Claim 17. Let $a \in \{1, 2, \dots, b\}$. Beginning with vertex 1, the first edge in T is the base chord $\{1, a + 2\}$ and the rest of the edges follow the path $a + 2, a + 2 - 1, a + 2 - 2, \dots, 2$. Clearly path T has a length of $a + 1$. □

Since path T is only in $G_b(1, 0)$ and path S does not use any chord edges in $F_k(n)$ nor does S contain the first $b + 1$ outer edges, we know that S and T are edge disjoint. By joining paths T and S in a union, $T \cup S$, we create cycles of length $n - b + a$. Because $n > k^k$, we know that $b > k$, therefore cycles of length $l \in \{n - k + 1, n - k + 2, \dots, n\}$ are contained within $n - b + a$ where $a \in \{1, 2, \dots, b\}$. By Lemma 8 and Lemma 15, $D_k(n)$ contains a cycle of length l for each $l \in \{k, k + 1, k + 2, \dots, n\}$. □

3 Concluding Remarks and Acknowledgments

We believe that our upper bound is asymptotically best possible and that the coefficient of $n^{1/k}$ in the bound $c(n, k) \leq k \lceil n^{1/k} \rceil + k$ is correct.

It is likely that the assumption that $n \geq (k + 2)^k$ is not needed. For smaller n , the construction is similar except for the fact that we would have to deal with several non-trivial induced subgraphs of the general $G_b(i, e)$. These subgraphs would vary highly depending on n .

Chaouche et. al. ask if $c(n, k)$ is monotone in n . We believe that $c(n, k)$ is monotone in n for all $n > (k + 2)^k$. Establishing monotonicity for these types of problems seems difficult. For example, Griffin [5] has conjectured that the function $m(n)$, defined in the introduction, satisfies $m(n) \leq m(n + 1)$ for all $n \geq 3$, but to our knowledge, this is still open.

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